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Proposed Method for Modeling the Gravitational Interaction Between Finite Bodies

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A new computational approach to the problem of the mutual gravitational attraction between finite-size bodies is presented. In this case, the gravitational force acting on each body consists of contributions from the mutual interactions between all particles comprising all bodies. The current approach is mostly suitable for high-precision modeling of the gravitational perturbations acting on a satellite in the vicinity of large and irregular objects. This approach is preferable in problems such as a satellite in the vicinity of an asteroid or near a planetary ring. If the shape of the asteroid is irregular, the traditional Legendre polynomials method does not give a good approximation. The series expansion of the potential in the planetary ring problem has a convergence problem because the distance between the satellite and the ring center of mass can be small compared with the size of the ring. These difficulties are overcome by defining new integrals over the attracting body. The operations on these integrals for the purpose of methodical computation are also presented. The potential via radial basis functions is modeled, and the gradients based on these smooth functions are derived.

Nomenclature

$\mathrm{card}(\Sigma)$	=	number of elements is the set Σ (cardinality)
$\mathrm{com}(f)$	=	CPU time to compute f (time complexity)
$\boldsymbol{F}$	=	gravitational force
G	=	gravitational constant
K^h_{ijk}	=	integral over the outer body expressed in outer-body reference frame, (K integral); see Eq. (6)
M	=	mass
$\boldsymbol{R}$	=	radius vector between satellite c.m. and outer body mass element
$\Re_B$	=	satellite reference frame attached to principal axes
$\Re_O$	=	outer body reference frame
$\boldsymbol{r}$	=	radius vector between mass elements
$\boldsymbol{T}$	=	gravitational torque
${}^B T^O$	=	rotation matrix between $\Re_O$ and $\Re_B$
U^s_{pqr}	=	integral over the outer body expressed in satellite body reference frame, (U integral); see Eq. (4)
V	=	gravitational potential
X, Y, Z	=	coordinates in $\Re_O$ fixed to the outer body
X', Y', Z'	=	coordinates in $\Re_O$ with origin at satellite c.m.
x, y, z	=	coordinates in $\Re_B$ with origin at satellite c.m.
$\boldsymbol{\xi}$	=	radius vector between a fixed point in the outer body and the satellite c.m.

Subscripts

B	=	satellite body
O	=	outer body

I. Introduction

THE issue of mutual gravitational interaction between finite bodies is important when the bodies of interest are in relative proximity. In this case, one cannot model the attracting body as a point mass. Moreover, the corresponding gravitational potential consists of simultaneous interaction between all mass elements of the bodies. Representative examples are a satellite in the vicinity of an asteroid or a space station or a satellite perturbed by a planetary ring. The current research was motivated by a test of the equivalence principle in gravitational relativity.[1] The test configuration consists of two bodies, in free fall, inside a cylindrical capsule at high altitude. A null relative displacement between these masses would verify the equivalence principle. This experiment required a highly precise model of the gravitational interaction between the test masses and the capsule. This problem motivated the development of a new solution method, presented here. Although motivated by this specific problem, the method is general and can be useful in a variety of aerospace applications. In the following derivation, the body of interest will be denoted as the satellite, and the other bodies will be denoted as the outer body. The topology of this problem prevents the analyst from applying the common practice of expanding the outer body potential in an asymptotic series, as done in the traditional solutions.[2–4]

Previously published methods can be categorized by the level of approximation they employ. The simplest one is a finite body attracted by a point mass or by a body possessing spherical symmetry.[2] In this case the only integration is over the satellite body. A more involved approach is to model the outer body as a finite body, with the presumption that its size is small compared with the distance between the body and the satellite. This approach[3] enables the asymptotic expansion of the potential over the two bodies. The near spherical symmetry of planets motivated a spherical coordinate representation,[4] whereas the arbitrary shapes of artificial satellites motivated a Cartesian representation.[2,3] A further complication is when the outer body is topologically unfit for an asymptotic expansion, as in the current problem. This research avoids a brute-force numerical solution consisting of six-dimensional volume integration and aims for a simple formulation and efficient algorithm. The efficiency is important for satellite dynamics problems, where the mutual gravitational perturbations should be computed for a large set of positions and attitudes, at each time step.

The current solution consists of two phases: a gravitational model and a computational model. The first phase presented in Sec. II consists of a formulation of integrals and operations on these integrals for defining the potential and its derivatives. The second phase presented in Sec. III consists of model reduction via presentation of the potential in terms of radial basis functions.

II. Model for Mutual Gravitational Attraction

Given a satellite and an outer body, our goal is to compute the gravitational perturbations acting on the satellite. The mutual

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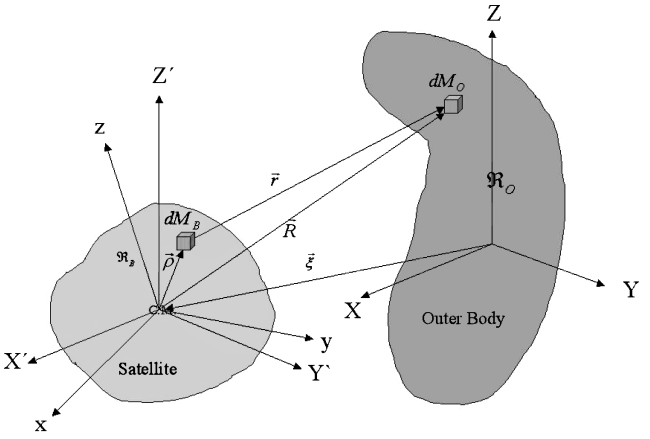

Fig. 1 Geometry of the model.

gravitational potential is

$$V = -G \iiint_O \iiint_B \frac{1}{r}\, \mathrm{d}M_B\, \mathrm{d}M_O \tag{1}$$

The integration procedure starts with integration over the satellite. The term $1/r$ is expanded in R, ρ assuming that $\rho < R$, as described in Fig. 1. The integration over the satellite, with R, ρ projected on the body frame, produces a series in powers of $1/R$ with the inertia integrals[2]:

$$J_{\underbrace{x\cdots x}_{p\text{-times}}\ \underbrace{y\cdots y}_{q\text{-times}}\ \underbrace{z\cdots z}_{r\text{-times}}} = \iiint_B x^p y^q z^r\, \mathrm{d}M_B \tag{2}$$

For clarity, this presentation retains only the zeroth-order potential (power of $1/R$) and the second-order potential (power of $1/R^3$). Higher-order expansion is straightforward,[1] and it is not presented in order to keep the presentation short. In most applications the computation of the first leading terms is sufficient. However, in some applications the higher-order terms are significant. For example, Ref. 1 is concerned with the order of $1/R^5$ because, as a result of a specific design, all of the lower terms are null. The next integration is over the outer body. Unfortunately, the convergence of the asymptotic series is not guaranteed for an arbitrary topology of the bodies. The following technique deals with this limitation.

The truncated potential after omitting higher-order terms is

$$V = -G \iiint_O \left\{ \frac{M_B}{R} + \frac{1}{2R^3}\left[(3l^2-1)J_{xx} + (3m^2-1)J_{yy} + (3n^2-1)J_{zz}\right] + H.O.T \right\} \mathrm{d}M_O \tag{3}$$

where J_{xx}, J_{yy}, J_{zz} are the second-order inertia integrals, and l, m, n are the direction cosines: $l = x/R$, $m = y/R$, $n = z/R$. These direction cosines reflect the orientation of an outer-body mass element with respect to the satellite reference frame. Note that the order of $1/R^2$ is null because the origin coincides with the center of mass.

A straightforward integration over $\Re_O$ includes the integration of algebraic terms of the form $x^p y^q z^r / R^s$. Note that $s - p - q - r = \deg(J) + 1$, where $\deg(J)$ is the order of the inertial integral of a particular term in the potential expansion. The integrals of these terms expressed in $\Re_B$ are defined as the U integrals:

$$U^s_{pqr} = \iiint_O \frac{x^p y^q z^r}{R^s}\, \mathrm{d}M_O \tag{4}$$

Expressing the direction cosines of Eq. (3) in terms of x, y, z and substituting Eq. (4) results in the following potential (orders 0 and 2 are presented) in terms of the U integrals:

$$V = -G\left\{ M_B U^1_{000} + \tfrac{1}{2}\left[\left(3U^5_{200} - U^3_{000}\right)J_{xx} + \left(3U^5_{020} - U^3_{000}\right)J_{yy} + \left(3U^5_{002} - U^3_{000}\right)J_{zz} \right] \right\} \tag{5}$$

Note that there are two frames of references: $\Re_B$ is attached to the satellite principal axes, and $\Re_O$ is attached to the outer body. Accordingly, x, y, z are coordinates in $\Re_B$ with origin at satellite c.m., and X, Y, Z are coordinates in $\Re_O$ fixed to the outer body. We define also X', Y', Z' as coordinates in $\Re_O$ with origin at the satellite c.m. The coordinate system X', Y', Z' is related to the coordinate system X, Y, Z by a pure translation, and the coordinate system x, y, z is related to the coordinate system X', Y', Z' by a pure rotation. Note that the origin of both coordinate systems x, y, z and X', Y', Z' coincides with the satellite c.m. while the origin of X, Y, Z is arbitrary (although it is convenient to attach it to the outer-body c.m.). These integrals depend on the six-dimensional configuration space (relative position and relative orientation). It is more efficient to express the U integrals in terms of integrals that are independent of the relative orientation. Consequently, the following integrals are defined as the K integrals:

$$K^h_{ijk} = \iiint_O \frac{(X')^i (Y')^j (Z')^k}{R^h}\, \mathrm{d}M_O \tag{6}$$

Because the integration is now in the outer-body reference frame, the K integrals depend on the position of the satellite but not on its orientation. In other words, the U integral has a value for each position and orientation of the satellite (relative to the outer body). On the other hand, the K integral has a value for each satellite position regardless of the satellite orientation. A U integral is a linear combination of K integrals, as shown by Eq. (8). Each K integral defines a field in $\Re_O$. It is rare to find a closed-form solution, and so the integral is usually evaluated by numerical integration. Both U and K integrals will be denoted as the outer integrals.

The U integrals are related to the K integrals through the transformation

$$[x\ y\ z]^T = {}^B T^O [X'\ Y'\ Z']^T \tag{7}$$

where ${}^B T^O$ is the rotation matrix between $\Re_O$ and $\Re_B$. A substitution into Eq. (4) reveals that the relation between the U integrals and the K integrals is

$$U^s_{pqr} = \iiint_O \frac{1}{R^s}(T_{11}X' + T_{12}Y' + T_{13}Z')^p (T_{21}X' + T_{22}Y' + T_{23}Z')^q \times (T_{31}X' + T_{32}Y' + T_{33}Z')^r\, \mathrm{d}M_O \tag{8}$$

where T_{ij} are the components of the rotation matrix ${}^B T^O$. The integrand can be expanded to Cartesian products with coefficients that are functions of the orientation. The integral of each Cartesian product is a K integral. Note that each U^s_{pqr} is a linear combination of $\frac{1}{2}(p+q+r+1)(p+q+r+2)$ K integrals.

Consequently, the potential in terms of the K integrals is

$$V = -G\left[M_B K^1_{000} + \tfrac{3}{2}(Jxx\boldsymbol{C}_1 + Jyy\boldsymbol{C}_2 + Jzz\boldsymbol{C}_3)\boldsymbol{K}_2 \right] \tag{9}$$

where $\boldsymbol{K}_2$ is a vector of the second-order K integrals and $\boldsymbol{C}_j$ are vector functions of the orientation:

$$\boldsymbol{K}_2 = \begin{bmatrix} K^5_{200} \\ K^5_{020} \\ K^5_{002} \\ K^5_{110} \\ K^5_{101} \\ K^5_{011} \\ K^3_{000} \end{bmatrix}, \qquad \boldsymbol{C}_1 = \begin{bmatrix} T^2_{11} \\ T^2_{12} \\ T^2_{13} \\ 2T_{11}T_{12} \\ 2T_{11}T_{13} \\ 2T_{12}T_{13} \\ -\frac{1}{3} \end{bmatrix}$$

$$\boldsymbol{C}_1 = \begin{bmatrix} T^2_{21} \\ T^2_{22} \\ T^2_{23} \\ 2T_{21}T_{22} \\ 2T_{21}T_{23} \\ 2T_{22}T_{23} \\ -\frac{1}{3} \end{bmatrix}, \qquad \boldsymbol{C}_1 = \begin{bmatrix} T^2_{31} \\ T^2_{32} \\ T^2_{33} \\ 2T_{31}T_{32} \\ 2T_{31}T_{33} \\ 2T_{32}T_{33} \\ -\frac{1}{3} \end{bmatrix} \tag{10}$$

The force acting on the satellite can be derived from both U and K representations. However, it is more efficient to derive it as a gradient of the U potential and then to expand it in K integrals. For that purpose, gradients of the U integrals can be expressed in terms of other U integrals:

$$\nabla_r U^h_{ijk} = \begin{bmatrix} iU^h_{i-1jk} - hU^{h+2}_{i+1jk} \\ jU^h_{ij-1k} - hU^{h+2}_{ij+1k} \\ kU^h_{ijk-1} - hU^{h+2}_{ijk+1} \end{bmatrix} \tag{11}$$

Therefore, the corresponding force is

$$F_x = \frac{\partial V}{\partial x} = G\left\{ M_B U^3_{100} + \frac{3}{2}\left[\left(5U^7_{300} - 3U^5_{100}\right) J_{xx} \right.\right.$$
$$\left.\left. + \left(5U^7_{120} - U^5_{100}\right) J_{yy} + \left(5U^7_{102} - U^5_{100}\right) J_{zz} \right] \right\}$$

$$F_y = \frac{\partial V}{\partial y} = G\left\{ M_B U^3_{010} + \frac{3}{2}\left[\left(5U^7_{210} - U^5_{010}\right) J_{xx} \right.\right.$$
$$\left.\left. + \left(5U^7_{030} - 3U^5_{010}\right) J_{yy} + \left(5U^7_{012} - U^5_{010}\right) J_{zz} \right] \right\}$$

$$F_z = \frac{\partial V}{\partial z} = G\left\{ M_B U^3_{001} + \frac{3}{2}\left[\left(5U^7_{201} - U^5_{001}\right) J_{xx} \right.\right.$$
$$\left.\left. + \left(5U^7_{021} - U^5_{001}\right) J_{yy} + \left(5U^7_{003} - 3U^5_{001}\right) J_{zz} \right] \right\} \tag{12}$$

Expanding the U integrals produces 16 K integrals: $K^3_{100}, K^3_{010}, K^3_{001}$ from the zero-order term and $K^5_{100}, K^5_{010}, K^5_{001}, K^7_{210}, K^7_{201}, K^7_{021}, K^7_{012}, K^7_{102}, K^7_{012}, K^7_{111}, K^7_{300}, K^7_{030}, K^7_{003}$ from the second-order term. Similarly, the torque in terms of the U integrals is

$$T_x = 3G(Jyy - Jzz)U^5_{011}, \qquad T_y = 3G(Jzz - Jxx)U^5_{101}$$
$$T_z = 3G(Jxx - Jyy)U^5_{110} \tag{13}$$

The U integrals of the torque contain six K integrals: K^5_{200}, K^5_{020}, K^5_{002}, K^5_{110}, K^5_{101}, and K^5_{011}. Note that these are exactly the six K integrals of the second-order potential.

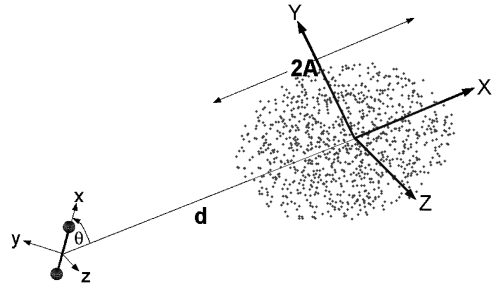

Fig. 2 Dumbbell satellite in the vicinity of an outer body.

In summary, the direct computation of the froce/torque requires the evaluation of 22 K integrals over the outer body, for each satellite location.

The following problem was chosen to verify the proposed method. The model for the outer body consists of 100 mass points bounded inside an ellipsoid. The mass points of this artificial body are randomly distributed with a uniform distribution. The satellite is represented as an ideal dumbbell (two mass points). The dumbbell is oriented by a pitch angle ϑ with respect to its z axis as illustrated in Fig. 2. The advantage of this type of a test bed is that the brute-force numerical solution is the exact solution. The model for the outer body is not a discretization, and so we avoided an additional source of error. Thus, the numerical potential is identical with the exact analytical potential. For methodical purpose, the units are nondimensional. The gravitational constant G and the length and the mass of the satellite are taken as one, and thus the potential is nondimensional. The size of the ellipsoid is $A = 0.5$, $B = 0.4$, and $C = 0.3$, where A, B, and C are the semiaxes of the ellipsoid. The ratio between the mass of the satellite and the mass of the outer body is 1:10.

Figure 3 shows the potential as a function of the normalized distance between the centers of mass d/A (see Fig. 2). The pitch angle for this plot is $\vartheta = 45$ deg. The exact potential was computed by summing all combinations of interactions between the dumbbell mass elements m_i and the outer-body mass elements m_j, that is,

$$V = -G \sum_{i=1}^{N_S} \sum_{j=1}^{N_B} \frac{1}{r_{ij}} m_i m_j \tag{14}$$

where r_{ij} is the distance between m_i and m_j and N_S and N_B are the number of the satellite mass elements and the outer-body mass elements, respectively. There is close agreement with the potential according to the current method, denoted as the k potential. The concept of the radial basis function (RBF) potential is explained in the next section.

Figure 4 shows the pitch torque acting on the dumbbell, located at $\mathrm{d}/A = 1$. The exact torque is computed by summing the individual contributions from each mass element:

$$\boldsymbol{T} = -G \sum_{i=1}^{N_S} \sum_{j=1}^{N_B} \frac{\boldsymbol{\rho}_i \times \boldsymbol{R}_j}{r^3_{ij}} m_i m_j \tag{15}$$

where ρ_i is a radius vector between the satellite c.m. and m_i and R_j is a radius vector between the satellite c.m. and m_j. The exact torque and the torque according to the current method, both in a good agreement, are compared with the classical MacCullagh torque. The latter is the leading term of the asymptotic expansion of the torque acting on a finite body, attracted by a point mass. Note that the maximum relative error between the K torque and the exact torque is 0.5%, while the maximum relative error of the MacCullagh torque is 15%. The deviation between the MacCullagh torque and the two finite bodies' torque is diminished when the bodies are far apart. It is expected that the deviation from the MacCullagh formulation will become wider for irregular-shaped outer bodies, for instance, a satellite near a planetary ring. Thus the advantage of the current method

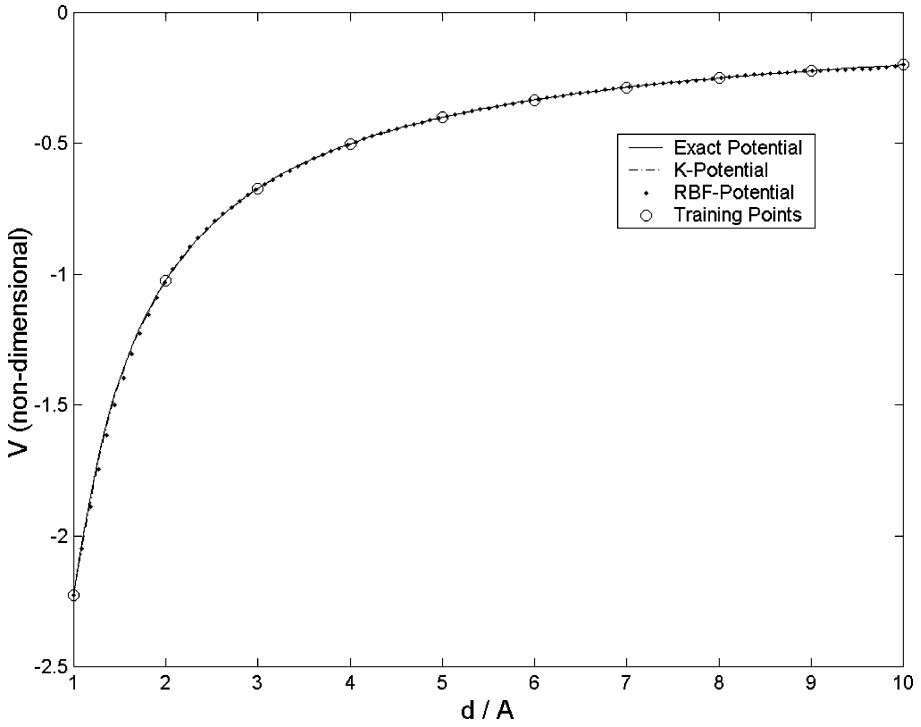

Fig. 3 Potential V as a function of the normalized mutual distance between the centers of the mass d/A.

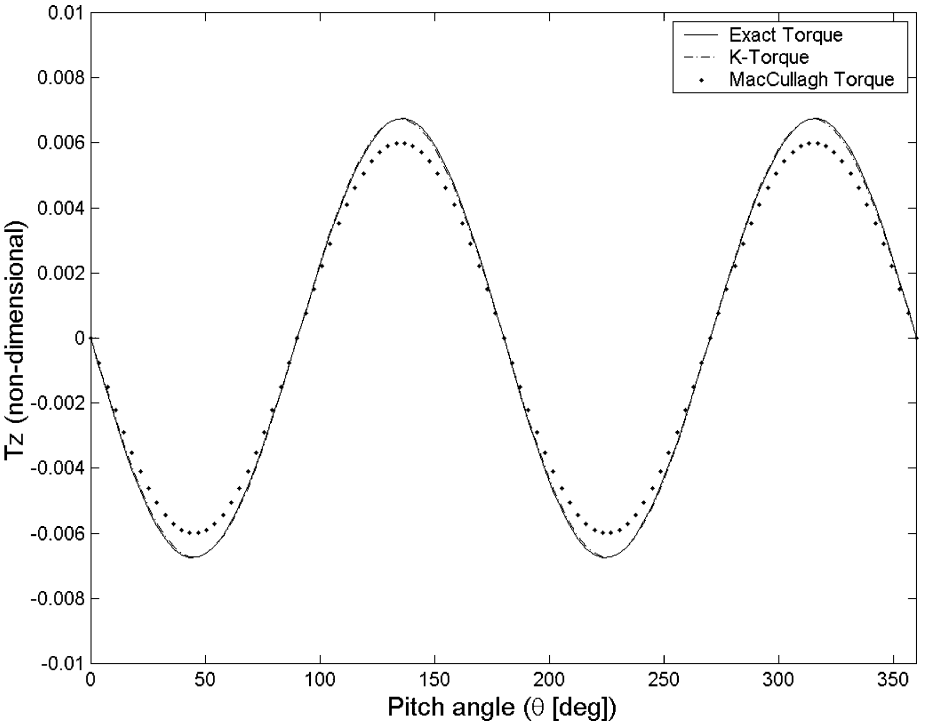

Fig. 4 Torque T_z as a function of the dumbbell pitch angle ϑ.

is for high-precision computation when the bodies of concern are in proximity and when the outer body has an irregular shape.

III. Mutual Gravitational Attraction in Terms of Radial Basis Functions

The direct computation of the force/torque requires a significant numerical integration of numerous K integrals over the outer bodies. This complexity motivated us to seek a model reduction that preserves precision and smoothness. Research on the approximation of gravitational field models started with pioneering works[5,6] aimed at the geopotential. These works used various weighting functions to approximate an a priori determined gravitational model. A promising tool for this purpose that is well suited to the current model consists of the RBFs. RBFs have been used in neural-networks applications[7–9] and have been proven to be a universal approximation[10] of continuous surfaces of any dimension. RBFs are widely used now in computer graphics, especially image synthesis, morphing technology, facial animation, and engineering drawing.[11,12] In general, RBFs are useful in reconstructing smooth

surfaces from scattered points. In other words, the RBFs offer a compact functional description of a set of surface data. The data are not required to lie on a regular grid or to be uniformly sampled. This advantage is appealing for modeling the gravitational fields associated with irregular-shaped bodies. The desired approximation in our case is a mapping from the three-dimensional configuration space to a K integral. The RBF approximation is a weighted sum of translations of radial symmetric basic functions (kernels). The kernel used in the current work is Gaussian:

$$\varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n) = \exp\left(-\frac{\|\boldsymbol{\xi} - \boldsymbol{\xi}_n\|_2}{2\sigma^2}\right) \tag{16}$$

$\boldsymbol{\xi}$ and $\boldsymbol{\xi}_n$ represent the radius vector of a fitting (evaluation) point and a radius vector of a training point, respectively. The training points represent a given data used to compute the RBF's weights. An evaluation point is a desired point for interpolation. $\boldsymbol{\xi}$ is a vector attached to $\Re_O$. The K integral should be computed (usually by numerical integration) at each training point. The RBF approximation for a particular K integral becomes

$$K^l_{pqr}(\boldsymbol{\xi}) = \sum_{n=1}^{N} \left(W^l_{pqr}\right)_n \varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n) \tag{17}$$

where W^l_{pqr} are the unknown weights. The RBF approximation is local in the sense that each RBF gives an exact solution on a training point, but the solution fitting diminishes with the distance from the training point. The locality of the fitting is controlled by σ (variance like). A small σ produces good local fit, but "noisy" approximation otherwise (overfit). A large σ produces better smoothing but a worse local approximation. The best σ is a tradeoff between local approximation and smoothing and is proportional to the space between the grid points. To compute the weights, the K integrals are evaluated on a set of N training points, and this generates the training data (or target data). The basic approach is to form a set of N equations of the form

$$\underbrace{\begin{bmatrix} \phi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_1) & \cdots & \phi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_N) \\ \vdots & \ddots & \vdots \\ \phi(\boldsymbol{\xi}_N, \boldsymbol{\xi}_1) & \cdots & \phi(\boldsymbol{\xi}_N, \boldsymbol{\xi}_N) \end{bmatrix}}_{\boldsymbol{\Phi}} \boldsymbol{W}^l_{\mathrm{pqr}} = \boldsymbol{K}^l_{\mathrm{pqr}} \tag{18}$$

where $\boldsymbol{W}^l_{pqr} = [(W^l_{pqr})_1 \cdots (W^l_{pqr})_N]^T$ is an unknown weights vector and $\boldsymbol{K}^l_{pqr} = [K^l_{pqr}(\boldsymbol{\xi}_1) \cdots K^l_{pqr}(\boldsymbol{\xi}_N)]^T$ is the training vector of K^l_{pqr}. The computation of the K integrals is a source of significant computational effort, although there are few cases where the K integrals have an analytical solution.[1] One way to simplify the computation is to approximate the K integrals in the case that the bodies are far apart. In this case the K integrals degenerate into inertia integrals of the outer body. However this approximation is unfit for the type of problems presented in this paper. The RBF's approximation thus reduces the computational load. Given the computed K integrals at the training points, the solution is simply $\boldsymbol{W}^l_{pqr} = \Phi^{-1}\boldsymbol{K}^l_{pqr}$. Note that the matrix Φ is symmetric. In addition, the diagonal of the matrix consists of ones because $\Phi(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) = 1$. There are variations of this algorithm, such as regularization[9] and fast RBF.[13] In addition to the current example, the method presented was tested for the gravitational field inside a cylinder and showed a smooth fit with high accuracy.[1]

A. Gravitational Potential and Its Derivatives in Terms of RBFs

If each of the K integrals is expanded in RBFs, then the second-order potential becomes

$$V(\boldsymbol{\xi}) = -G \sum_{n=1}^{N} \left[M_B \left(W^1_{000}\right)_n + \frac{3}{2}(Jxx\boldsymbol{C}_1 + Jyy\boldsymbol{C}_2 + Jzz\boldsymbol{C}_3)(\boldsymbol{W}_2)_n \right] \varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n) \tag{19}$$

where $(\boldsymbol{W}_2)_n = [(W^5_{200})_n (W^5_{020})_n (W^5_{002})_n (W^5_{110})_n (W^5_{101})_n (W^5_{011})_n (W^3_{000})_n]^T$ and $\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3$ are vector functions of the attitude transformation. The fitting is demonstrated in Fig. 3. There were 10 training points, and the variance was 1.5 units. A remarkable advantage of the preceding formulation is the small number of K integrals: eight integrals instead of 22 integrals needed for a direct computation of the equivalent force/torque. The force in $\Re_B$ is the gradient of the RBF potential expressed in $\Re_B$:

$$\boldsymbol{F}(\boldsymbol{\xi}) = -G \sum_{n=1}^{N} \left[M_B \left(W^1_{000}\right)_n + \frac{3}{2}(Jxx\boldsymbol{C}_1 + Jyy\boldsymbol{C}_2 + Jzz\boldsymbol{C}_3)(\boldsymbol{W}_2)_n \right] \nabla\varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n)^O T^B \tag{20}$$

where

$$\nabla\varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n) = -(1/\sigma^2)[\boldsymbol{\xi} - \boldsymbol{\xi}_n]^T \varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n) \tag{20a}$$

Note that the gradient of the potential does not produce additional K integrals. Another nice feature is that the derivative of an RBF is another RBF. Therefore, the RBF representation is convenient for further derivations. For example, it is easy to derive the gravity gradient, which is the second-order tensor of the potential. Furthermore, the gradient is a very smooth function (contrary to derivatives of a polynomial approximation). Numerical tests show that for sufficient evaluations the derivatives of RBF's fitting retain the desired precision.

Because the six K integrals needed for the torque are already computed for the second-order potential, a simple substitution gives the torque in terms of RBFs:

$$\begin{aligned}
Tx = 3G(Jyy - Jzz) \sum_{n=1}^{N} \Big[& T_{21}T_{31}\left(W^5_{200}\right)_n + T_{22}T_{32}\left(W^5_{020}\right)_n \\
& + T_{23}T_{33}\left(W^5_{002}\right)_n + (T_{22}T_{31} + T_{21}T_{32})\left(W^5_{110}\right)_n + (T_{23}T_{31} \\
& + T_{21}T_{33})\left(W^5_{101}\right)_n + (T_{23}T_{32} + T_{22}T_{33})\left(W^5_{011}\right)_n \Big] \varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n) \\
Ty = 3G(Jzz - Jxx) \sum_{n=1}^{N} \Big[& T_{11}T_{31}\left(W^5_{200}\right)_n + T_{12}T_{32}\left(W^5_{020}\right)_n \\
& + T_{13}T_{33}\left(W^5_{002}\right)_n + (T_{12}T_{31} + T_{11}T_{32})\left(W^5_{110}\right)_n + (T_{13}T_{31} \\
& + T_{11}T_{33})\left(W^5_{101}\right)_n + (T_{13}T_{32} + T_{12}T_{33})\left(W^5_{011}\right)_n \Big] \varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n) \\
Tz = 3G(Jxx - Jyy) \sum_{n=1}^{N} \Big[& T_{11}T_{21}\left(W^5_{200}\right)_n + T_{12}T_{22}\left(W^5_{020}\right)_n \\
& + T_{13}T_{23}\left(W^5_{002}\right)_n + (T_{12}T_{21} + T_{11}T_{22})\left(W^5_{110}\right)_n + (T_{13}T_{21} \\
& + T_{11}T_{23})\left(W^5_{101}\right)_n + (T_{13}T_{22} + T_{12}T_{23})\left(W^5_{011}\right)_n \Big] \varphi(\boldsymbol{\xi}, \boldsymbol{\xi}_n)
\end{aligned} \tag{21}$$

The steps for computation are as follows:

1) Assign training points in the desired evaluation space. These training points are scattered around the expected location of the satellite. The density of the training points is determined by the required accuracy.

2) Evaluate the K integrals at these points according to Eq. (6) This evaluation is usually done by a numerical integration.

3) Compute the kernels [Eq. (16)], and construct the kernel matrix Φ [Eq. (18)]. Note that the number of RBFs is the number of training points.

4) Compute the RBF's weights by solving Eq. (18).

5) Compute the transformation matrix between the outer body and the satellite reference frames.

6) Evaluate the force/torque at the desired evaluation points using Eqs. (20) and (21). It is recommended to avoid large kernel matrices if one uses this basic RBF method. (Φ can be ill conditioned

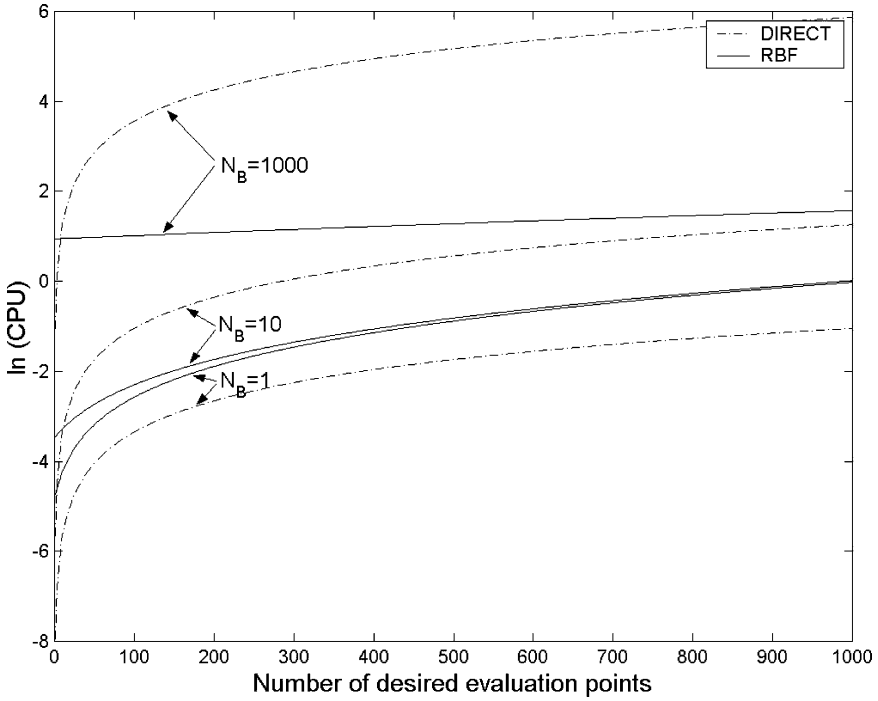

Fig. 5 Time complexity of the direct and the RBF computation vs the number of desired evaluation points, for various number of outer-body mass elements N_B.

when the training set is large.) One approach to overcome this difficulty is to divide the evaluation space to few subspaces and then to patch the evaluations. Another approach is to use a regularization method.[9]

B. Estimation of the Algorithm Complexity

Using the RBF method requires additional preliminary work, but it can increase the efficiency of the computation. The following discussion demonstrates a qualitative analysis on the complexity of the algorithms. In short, we will compare the required computational effort in both methods, in order to achieve a given precision. Assume that there is a set of points [evaluation set, Σ_E, $N_K = \text{card}(\Sigma_E)$] that covers the desired space and that the K integrals are computed on this set. The maximum relative error of a K integral at any other point in the space is denoted as ε. Define this property as ε compactness. The density of the set depends on the ε compactness and on the gradients of the gravity field. To guarantee the desired precision, the proper interval between two neighboring points on the set should be of order $\varepsilon K(r)/|\nabla_r K|$. This guarantees that the computation of a K integral at the evaluation points is sufficient for approximating the K integral at any other point, with the desired precision. Only a subset of points [training set Σ_T, $N_{\text{RBF}} = \text{card}(\Sigma_{\text{RBF}}) =$ the number of RBFs] is needed for training the RBF. The ratio between the cardinality of these two sets ($\gamma = N_{\text{RBF}}/N_k \ll 1$) depends on the dimension and on the topology of the space. In addition, the number of the K integrals to be evaluated using direct computation (NK) is higher than the number of K integrals to be evaluated using RBFs (NK_{RBF}). For instance, $NK_{\text{RBF}}/NK = 8/22$, given the truncated potential shown in the last section. The complexity of the direct computation is simply $N_K \cdot \text{com}(K) \cdot NK$, where $\text{com}(K)$ is the time complexity of computing a single K integral (the CPU time for the computation). It depends linearly on the number of mass elements of the outer body needed for the integration. The complexity of the RBF is a sum of three complexities caused by 1) evaluating the K integrals on the training set, 2) computing of the weights, and 3) evaluating the RBF on the evaluation set. The first complexity is $N_{\text{RBF}} \cdot \text{com}(K) \cdot NK_{\text{RBF}}$. The second complexity depends on N_{RBF} and on the efficiency of the algorithm. The complexity metric in the current work is the measured CPU of the weights computation as a function of N_{RBF}. The third complexity resembles the complexity of the direct computation, but the complexity of computing a single K integral depends now on the number of RBF weights instead of the number of mass elements on $\Re_O$. Thus, the third complexity is $N_k \cdot \text{com}[K(RBF)] \cdot NK_{\text{RBF}}$.

The model for estimating the complexity is the same as in the preceding example. The outer body was modeled as a random distribution of N_B mass points inside an ellipsoid. Note that the shape of the satellite does not affect the computation complexity, whereas the shape of the outer body has only a minor effect. Therefore, any other mass distribution would provide similar results. The computational complexities, direct and via RBFs, are illustrated in Fig. 5. The complexity metric in Fig. 5 is defined as the logarithm of the measured CPU time required to compute the K integrals in Eqs. (12) and (13). The independent variables are the number of evaluation points and the number of mass elements N_B. The RBF method is preferable when the number of evaluation points and the number of mass elements are large (and countereffective for a very simple outer body). Furthermore, the current RBF algorithm is basic, and there are faster algorithms.[13] Applying the fast RBF can reduce the complexity by a factor of $(1/N)\,\ell_n N$; thus, Fig. 5 presents a conservative prediction, and the expected performance of the RBF is clearly superior.

IV. Conclusions

This paper presents a new approach for calculating the mutual gravitational potential between arbitrary bodies. The model includes intertia integrals and the so-called outer integrals. The former are properties of the satellite, whereas the latter are fields related to the mass distribution of the outer body. This paper presents only a few terms of the potential, but it is straightforward to express any higher-order terms in terms of the outer integrals. The advantage of this method is for high-precision computation when the bodies of concern are in proximity and when the outer body has an irregular shape. Another technique is to formulate the potential in terms of radial basis functions (RBF) and then to derive the gradients. A study of the computational effort indicates that the RBF approximation of the potential is preferable for representing complex models. The method was validated numerically and found to be in a close agreement with the exact solution. Future research for refining the

latest method and embedding it with aerospace applications is recommended. Future research will seek to represent the potential by backpropagating a neural network and then to derive the gradients numerically.

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